ON FLOWS OF AN IDEAL GAS WHOSE SONIC SURFACE COINCIDES WITH A CHARACTERISTIC SURFACE

(O TECHENIIAKH IDEAL'NOGO GAZA SO ZVUKOVOI POVERKHNOST'IU, SOVPADAIUSHCHEI S KHARAKTERISTICHESKOI)

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E.G. SHIFRIN
(Moscow)

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We investigate the flow of an ideal gas whose sonic surface coincides with a characteristic one (if the flow exists). For brevity, we shall call it an A-flow. In [1] a general condition for A-flows was found, namely, the sonic surface is a minimal one. In the present paper, a supplementary, necessary condition for potential A-flows is presented; in particular, it is not satisfied in the axisymmetric case (*).

The theorem of [1] is deduced from the Bernoulli equations of continuity, of the state $p = p(\rho, S)$ at the condition of isentropic flow. If $\mathbf{n}_1$ is the unit velocity vector, then

$$\operatorname{div}(\rho v \mathbf{n}_1) = \frac{\partial \rho v}{\partial s_1} + \rho v \operatorname{div} \mathbf{n}_1 = 0, \qquad \frac{\partial}{\partial s_1} = \mathbf{n}_1 \cdot \nabla \tag{1}$$

Since $\partial(\rho v)/\partial s_1 = 0$ for $v = a$, it follows from (1) that $\operatorname{div} \mathbf{n}_1 = 0$ for $v = a$; this is the condition that the surface orthogonal to $\mathbf{n}_1$ is a minimal one.

Now let us investigate the equation of motion $(\mathbf{V} \cdot \nabla) \mathbf{V} + \rho^{-1} \nabla p = 0$, written in the form [2]

$$v \frac{\partial v}{\partial s_1} + \frac{1}{\rho} \frac{\partial p}{\partial s_1} = 0, \quad -v^2 \varkappa + \frac{1}{\rho} \frac{\partial p}{\partial s_2} = 0, \quad \frac{\partial p}{\partial s_3} = 0 \qquad \left(\frac{\partial}{\partial s_i} = \mathbf{n}_i \cdot \nabla\right) \tag{2}$$

Here $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors of the principal normal and the binormal to a streamline, $\varkappa \geqslant 0$ is the streamline curvature.

From the second equation of (2) it follows that, if there exists a potential A-flow, then streamlines in the neighborhood of the sonic surface are approximated by straight lines, to accuracy up to third-order small quantities.

We shall determine the class of three-dimensional flows for which this necessary condition can occur.

Introduce into the potential A-flow an oblique system of coordinates so that two families, u_2, u_3, are stream surfaces while the u_1 is orthogonal to streamlines. (The surfaces u_1 are assumed to be sufficiently smooth.)

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*) The fact that A-flows cannot exist in the axisymmetric case was pointed out to the author by Iu.D.Shmyglevskii (in searching for a solution of the Cauchy problem in the neighborhood of the sonic line in the form of a power series, the coefficients turned out to be imaginary).

Consider a twice continuously differentiable streamline $\mathbf{r}(u_1)$ which intersects the sonic surface. Construct a tetrahedral elementary stream tube with constant cross-section area ε so that two sides of the tube are defined by the surfaces u_2 and u_3, intersecting on the curve $\mathbf{r}(u_1)$, and a third side by the surface $u_3 + du_3$. Denote by

$$\frac{1}{H_3} = \frac{du_3}{|\nabla u_3|}$$

the distance on the normal between the surfaces u_3 and $u_3 + du_3$ along the curve $\mathbf{r}(u_1)$; let $\mathbf{t}$ be a unit vector defined on the curve $\mathbf{r}(u_1)$ and obtained by rotating the vector $\mathbf{n}_1$ through the angle $\frac{1}{2}\pi$ in the given direction in the plane which is tangent to u_3.

The equation of the curve $\mathbf{R}(u_1)$, which lies on the surface u_3 and is the edge of the elementary tube of constant cross section, may be written, with accuracy to quantities of order ε^2, in the form

$$\mathbf{R}(u_1) = \mathbf{r}(u_1) + \varepsilon H_3(u_1)\,\mathbf{t}(u_1)$$

The area of the elementary stream tube has a minimum at the sonic point. Therefore, the streamline $\mathbf{r}(u_1)$ and the streamline which passes through the same point on the sonic surface as does the curve $\mathbf{R}(u_1)$ lie on different sides of $\mathbf{R}(u_1)$; the curvature of every streamline is zero at the sonic point; consequently, it is necessary that, at the sonic point, the projection of the curve $\mathbf{R}(u_1)$ on the tangent plane to the surface u_3 should not be convex to the streamline $\mathbf{r}(u_1)$ for sufficiently small ε.

If the unit vector of the curve $\mathbf{R}(u_1)$ be denoted by $\mathbf{N}_1$, then this condition may be written as follows:

$$\frac{\partial \mathbf{N}_1}{\partial u_1}\cdot\mathbf{t} \leqslant 0 \qquad \text{for} \quad v = a \tag{3}$$

Denoting a derivative with respect to u_1 by a prime, we obtain

$$\mathbf{N}_1 = \frac{\mathbf{R}'}{|\mathbf{R}'|} = \frac{|\mathbf{r}'|\,\mathbf{n}_1 + \varepsilon H_3'\mathbf{t} + \varepsilon H_3\mathbf{t}'}{(|\mathbf{r}'|^2 + \varepsilon^2 H_3'^2 + \varepsilon^2 H_3^2\mathbf{t}'\cdot\mathbf{t}' + 2\varepsilon\,|\mathbf{r}'|\,H_3\mathbf{n}_1\cdot\mathbf{t})^{1/2}}$$

$$\mathbf{N}_1' = \frac{1}{|\mathbf{R}'|}\{|\mathbf{r}'|'\,\mathbf{n}_1 - |\mathbf{r}'|^2\varkappa\mathbf{n}_2 + H_3''\varepsilon\mathbf{t} + H_3\varepsilon\mathbf{t}'' + 2\varepsilon H_3'\mathbf{t}' -$$

$$- \frac{1}{|\mathbf{R}'|^2}\left[|\mathbf{r}'|\,|\mathbf{r}'|' + \frac{\varepsilon}{2}(\varepsilon H_3'^2 + \varepsilon H_3^2\mathbf{t}'\cdot\mathbf{t}' + 2\,|\mathbf{r}'|\,H_3\mathbf{n}_1\cdot\mathbf{t}')'\right](|\mathbf{r}'|\,\mathbf{n}_1 + H_3'\varepsilon\mathbf{t} + H_3\varepsilon\mathbf{t}')\}$$

We choose the family u_1 so that $|\mathbf{r}'|' = 0$ on the line $\mathbf{r}(u_1)$ at the sonic point (e.g. we put $u_1 = s_1$, where s_1 is the arc length along the curve $\mathbf{r}(u_1)$).

Let γ be the geodesic curvature, δ the relative twist of the curve $r(u_1)$ on the surface u_3; then [3] we have

$$\mathbf{t}''\cdot\mathbf{t} = -\mathbf{t}'\cdot\mathbf{t}' = -|\mathbf{r}'|^2(\gamma^2 + \delta^2)$$

and condition (3) may be transformed to the form (quantities of order ε^2 are neglected)

$$\frac{1}{|\nabla u_3|}\frac{\partial^2}{\partial s_1^2}|\nabla u_3| \leqslant \delta^2 \qquad \text{for} \quad v = a \tag{4}$$

In a potential A-flow, this condition holds for arbitrary choice of the family u_3.

If the flow is such that the coordinate system u_i may be chosen to be thrice orthogonal (such a system is unique, with accuracy to the order indicated, if the flow is not uniform and rectilinear), then condition (4) for the surfaces u_2 and u_3 can be written in the form

$$\frac{\partial^2}{\partial s_1^2}|\nabla u_i| \leqslant 0 \qquad \text{for} \quad v = a \tag{5}$$

In the case of axial symmetry, this condition is not satisfied, as long as the sonic surface is not a plane perpendicular to the axis. In fact, we choose some streamline $y = y(x)$ and determine $u_1 = x$ on it. With this,

$$|\mathbf{r}'|' = \frac{y'y''}{(1+y'^2)^{1/2}} = 0 \qquad \text{for} \quad v = a, \qquad |\nabla u_3| = \frac{1}{y}$$

and condition (5) will be written in the form

$$yy'' \geqslant 2y'^2 \qquad \text{for} \quad v = a$$

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